

C_0 SEMIGROUP AND LOCAL SPECTRAL THEORY

A. TAJMOUATI, M. KARMOUNI, H. BOUA, Z. AL-HOMIDI

ABSTRACT. In this paper, we studied some local spectral properties for a C_0 semigroup and its generator. Some stabilities results are also established.

1. INTRODUCTION

The semigroups can be used to solve a large class of problems commonly known as the Cauchy problem:

$$u'(t) = Au(t), t \geq 0, u(0) = u_0 \quad (1)$$

on a Banach space X . Here A is a given linear operator with domain $D(A)$ and the initial value u_0 . The solution of (1) will be given by $u(t) = T(t)u_0$ for an operator semigroup $(T(t))_{t \geq 0}$ on X . In this paper, We will focus on a special class of linear semigroups called C_0 semigroups which are semigroups of strongly continuous bounded linear operators. Precisely:

A one-parameter family $(T(t))_{t \geq 0}$ of operators on a Banach space X is called a C_0 -semigroup of operators or a strongly continuous semigroup of operators if:

- (1) $T(0) = I$.
- (2) $T(t + s) = T(t)T(s)$, $\forall t, s \geq 0$.
- (3) $\lim_{t \rightarrow 0} T(t)x = x$, $\forall x \in X$.

$(T(t))_{t \geq 0}$ has a unique infinitesimal generator A defined in domain $D(A)$ by:

$$D(A) = \{x \in X : \lim_{t \rightarrow 0} \frac{T(t)x - x}{t} \text{ exists}\}$$

,

$$Ax = \lim_{t \rightarrow 0} \frac{T(t)x - x}{t}, \forall x \in D(A)$$

Recall that for all $t \geq 0$, $T(t)$ is a bounded linear operator on X and A is a closed operator. Details for all this may be found in [8, 4]. There are enough studies done on semi-groups, including spectral studies [1, ?, 2, 3, 4, 8, 9]. In this article, we investigate the transmission of some local spectral properties from a C_0 semi-group to its infinitesimal generator.

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2. PRELIMINARIES

Throughout, X denotes a complex Banach space, let A be a closed linear operator on X with domain $D(A)$, we denote by A^* , $R(A)$, $N(A)$, $R^\infty(A) = \bigcap_{n \geq 0} R(A^n)$, $\sigma_K(A)$, $\sigma_{su}(A)$, $\sigma(A)$, respectively the adjoint, the range, the null space, the hyper-range, the semi-regular spectrum, the surjectivity spectrum and the spectrum of A .

Recall that for a closed linear operator A and $x \in X$ the local resolvent of A at x , $\rho_A(x)$ defined as the union of all open subset U of \mathbb{C} for which there is an analytic function $f : U \rightarrow D(A)$ such that the equation $(A - \mu I)f(\mu) = x$ holds for all $\mu \in U$. The local spectrum $\sigma_A(x)$ of A at x is defined as $\sigma_A(x) = \mathbb{C} \setminus \rho_A(x)$. Evidently $\sigma_A(x) \subseteq \sigma_{su}(A) \subseteq \sigma(A)$, $\rho_A(x)$ is open and $\sigma_A(x)$ is closed.

If $f(z) = \sum_{i=0}^{\infty} x_i(z - \mu)^i$ (in a neighborhood of μ), be the Taylor expansion of f , it is easy to see that $\mu \in \rho_A(x)$ if and only if there exists a sequence $(x_i)_{i \geq 0} \subseteq D(A)$, $(A - \mu)x_0 = x$, $(A - \mu)x_{i+1} = x_i$, and $\sup_i \|x_i\|^{\frac{1}{i}} < \infty$, see [5, 7]. The local spectral subspace of A associated with a subset Ω of \mathbb{C} is the set :

$$X_A(\Omega) = \{x \in X : \sigma_A(x) \subseteq \Omega\}$$

Evidently $X_A(\Omega)$ is a hyperinvariant subspace of A not always closed.

Next, let A a closed linear operator, A is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP) if for every open neighborhood $U \subseteq \mathbb{C}$ of λ_0 , the only analytic function $f : U \rightarrow D(A)$ which satisfies the equation $(A - zI)f(z) = 0$ for all $z \in U$ is the function $f \equiv 0$. A is said to have the SVEP if A has the SVEP for every $\lambda \in \mathbb{C}$. Denote by

$$S(A) = \{\lambda \in \mathbb{C} : A \text{ has not the SVEP at } \lambda\}.$$

$S(A)$ is an open of \mathbb{C} and $X_A(\emptyset) = \{0\}$ implies $S(A) = \emptyset$ [1]. If A is bounded, then $X_A(\emptyset)$ is closed if and only if $X_A(\emptyset) = \{0\}$ if and only if $S(A) = \emptyset$ [6]. Note that $\mu \in S(A)$ if and only if there exists a sequence $(x_i)_{i \geq 0} \subseteq D(A)$ not all of them equal to zero such that $(A - \mu)x_{i+1} = x_i$, with $x_0 = 0$ and $\sup_i \|x_i\|^{\frac{1}{i}} < \infty$, see [5].

For a closed linear operator A the algebraic core $C(A)$ for A is the greatest subspace M of X for which $A(M \cap D(A)) = M$. Equivalently:

$$C(A) = \{x \in D(A) : \exists (x_n)_{n \geq 0} \subset D(A), \text{ such that } x_0 = x, Ax_n = x_{n-1} \text{ for all } n \geq 1\}$$

Moreover the analytic core for A is a linear subspace of X defined by:

$$K(A) = \{x \in D(A) : \exists (x_n)_{n \geq 0} \subset D(A) \text{ and } \delta > 0 \text{ such that } x_0 = x, Ax_n = x_{n-1} \quad \forall n \geq 1 \text{ and } \|x_n\| \leq \delta^n \|x\|\}$$

The analytica core admits a local spectral characterization for unbounded operator as follow [1, Theorem 4.3]:

$$K(A) = \{x \in D(A) : 0 \in \rho_A(x)\} = X_A(\mathbb{C} \setminus \{0\})$$

Note that in general neither $K(T)$ nor $C(T)$ are closed and we have

$$X_A(\emptyset) \subset K(A) \subseteq C(A) \subset R^\infty(A) \subset R(A).$$

Let $(T(t))_{t \geq 0}$, a C_0 semigroup with infinitesimal generator A , we introduce the following operator acting on X and depending on the parameters $\lambda \in \mathbb{C}$ and $t \geq 0$:

$$B_\lambda(t)x = \int_0^t e^{\lambda(t-s)}T(s)x ds, x \in X$$

It is well known that $B_\lambda(t)$ is a bounded linear operator on X [8, 4] and we have:

$$\begin{aligned} (e^{\lambda t} - T(t))^n x &= (\lambda - A)^n B_\lambda^n(t)x, \quad \forall x \in X, n \in \mathbb{N} \\ (e^{\lambda t} - T(t))^n x &= B_\lambda^n(t)(\lambda - A)^n x, \quad \forall x \in D(A^n), n \in \mathbb{N}; \\ R^\infty(e^{\lambda t} - T(t)) &\subseteq R^\infty(\lambda - A); \\ N((\lambda - A)^n) &\subseteq N(e^{\lambda t} - T(t))^n. \end{aligned}$$

In [8, 4], they showed that:

$$e^{t\nu(A)} \subseteq \nu(T(t))$$

where $\nu \in \{\sigma_p, \sigma_{ap}, \sigma_r\}$, point spectrum, approximative spectrum and residual spectrum.

In [2], the authors showed this spectral inclusion for the semi-regular spectrum.

In this work, as a continuous of the previous work, we will give a spectral inclusion for local spectrum. Also, we investigate some local spectral properties for C_0 semigroup and its generator. Some stabilities results are established.

3. LOCAL SPECTRAL THEORY

Now, we start the present section by the following lemma which we need in the sequel.

Lemma 3.1. [9] *Let $(T(t))_{t \geq 0}$ a C_0 -semigroup on X with infinitesimal generator A . For $\lambda \in \mathbb{C}$ and $t \geq 0$, let $F_\lambda(t)x = \int_0^t e^{-\lambda s} B_\lambda(s)x ds$, then:*

- (1) *There exist a $M \geq 1$ and $\omega > \operatorname{Re}(\lambda)$ such that $F_\lambda(t) \leq \frac{M}{(\omega - \operatorname{Re}(\lambda))^2} e^{(\omega - \operatorname{Re}(\lambda))t}$.*
- (2) *$\forall x \in X$, $F_\lambda(t)x \in D(A)$ and $(\lambda - A)F_\lambda(t) + G_\lambda(t)B_\lambda(t) = tI$ with $G_\lambda(t) = e^{-\lambda t}I$.*
- (3) *The operators $F_\lambda(t)$, $G_\lambda(t)$ and $B_\lambda(t)$ are pairwise commute and for all $x \in D(A)$:*

$$\begin{aligned} (\lambda - A)F_\lambda(t)x &= F_\lambda(t)(\lambda - A)x \\ (\lambda - A)G_\lambda(t)x &= G_\lambda(t)(\lambda - A)x \\ (\lambda - A)B_\lambda(t)x &= B_\lambda(t)(\lambda - A)x \end{aligned}$$

Theorem 3.1. *For the generator A of a strongly continuous semigroup $(T(t))_{t \geq 0}$, we have the following inclusion :*

$$S(T(t)) \subseteq e^{tS(A)}.$$

Proof. Suppose that $e^{\lambda t} - T(t)$ has not SVEP at 0, then there exist $x_i \in X$ such that $x_0 = 0$, $(e^{\lambda t} - T(t))x_i = x_{i-1}$ and $\sup_i \|x_i\|^{\frac{1}{i}} < \infty$. Let $y_i = B_\lambda^i(t)x_i$, then $(y_i)_{i \geq 0} \subseteq D(A)$ and $y_0 = x_0 = 0$, and we have :

$$\begin{aligned} (\lambda - A)y_i &= (\lambda - A)B_\lambda(t)B_\lambda^{i-1}(t)x_i \\ &= (e^{\lambda t} - T(t))B_\lambda^{i-1}(t)x_i \\ &= B_\lambda^{i-1}(t)(e^{\lambda t} - T(t))x_i \\ &= B_\lambda^{i-1}(t)x_{i-1} \\ &= y_{i-1} \end{aligned}$$

Therefore, $(\lambda - A)y_i = y_{i-1}$. On the other hand $\|y_i\| = \|B_\lambda^i(t)x_i\| \leq \|B_\lambda^i(t)\|\|x_i\| \leq M^i\|x_i\|$, then $\sup_i \|y_i\|^{\frac{1}{i}} \leq \sup_i M\|x_i\|^{\frac{1}{i}} < \infty$, $M > 0$. So that $\lambda - A$ has not SVEP at 0, then $S(T(t)) \subseteq e^{tS(A)}$. \square

In the following, we give a sufficient condition to show that the local spectral subspace $X_{T(t)}(\emptyset)$, $t > 0$, is closed.

Corollary 3.1. *Let $(T(t))_{t \geq 0}$ a C_0 -semigroup, with generator A . Then:*

$X_A(\emptyset) = \{0\}$ implies that $X_{T(t)}(\emptyset)$ is closed for all $t \geq 0$.

Proof. Since $X_A(\emptyset) = \{0\}$ implies that $S(A) = \emptyset$, by theorem 3.1 we have $S(T(t)) = \emptyset$ which equivalent to the fact that $X_{T(t)}(\emptyset) = \{0\}$ equivalently to $X_{T(t)}(\emptyset)$ is closed. \square

Corollary 3.2. *Let $(T(t))_{t \geq 0}$ a C_0 -semigroup, with generator A . If A has the SVEP then $T(t)$ has the SVEP for all $t \geq 0$.*

In the following Theorem, we give a spectral inclusion for the local spectrum.

Theorem 3.2. *Let $(T(t))_{t \geq 0}$ a C_0 -semigroup on X with infinitesimal generator A . The following spectral inclusion hold :*

$$e^{t\sigma_A(x)} \subseteq \sigma_{T(t)}(x)$$

Proof. Let $e^{\lambda t} \notin \sigma_{T(t)}(x)$, then there exists $(x_i)_{i \geq 0} \subseteq X$, such that

$$(e^{\lambda t} - T(t))x_0 = x, \quad (e^{\lambda t} - T(t))x_i = x_{i-1} \text{ and } \sup \|x_i\|^{\frac{1}{i}} < \infty.$$

Let $y_i = B_\lambda^{i+1}(t)x_i$, then $(y_i)_{i \geq 0} \subseteq D(A)$ and $y_0 = B_\lambda x_0$. We have :

$$\begin{aligned} (\lambda - A)y_i &= (\lambda - A)B_\lambda(t)B_\lambda^i(t)x_i \\ &= (e^{\lambda t} - T(t))B_\lambda^i(t)x_i \\ &= B_\lambda^i(t)(e^{\lambda t} - T(t))x_i \\ &= B_\lambda^i(t)x_{i-1} \\ &= y_{i-1} \end{aligned}$$

and

$$\sup \|y_i\|^{\frac{1}{i}} < \infty$$

So that $\lambda \notin \sigma_A(x)$

\square

Corollary 3.3. *Let $(T(t))_{t \geq 0}$ a C_0 -semigroup, with generator A . Then:*

- (1) $K(e^{\lambda t} - T(t)) \subseteq K(\lambda - A)$;
- (2) $C(e^{\lambda t} - T(t)) \subseteq C(\lambda - A)$.

Proof. Since $K(e^{\lambda t} - T(t)) = \{x \in X; 0 \in \rho_{(e^{\lambda t} - T(t))}(x)\}$, then if $x \in K(e^{\lambda t} - T(t))$ implies that $e^{\lambda t} \in \rho_{T(t)}(x)$, by theorem 3.2 we have $\lambda \in \rho_A(x)$, therefore $x \in K(\lambda - A)$. \square

Denote by $\sigma_{ac}(T) = \{\lambda \in \mathbb{C} : K(\lambda - T) = \{0\}\}$ the analytic core spectrum of T and by $\sigma_{alc}(T) = \{\lambda \in \mathbb{C} : C(\lambda - T) = \{0\}\}$ the algebraic core spectrum of T [10, 11]. As a straightforward consequence of the corollary 3.3, we have the following corollary.

Corollary 3.4. *Let $(T(t))_{t \geq 0}$ a C_0 -semigroup, with generator A . Then:*

- (1) $e^{t\sigma_{ac}(A)} \subseteq \sigma_{ac}(T(t));$
- (2) $e^{t\sigma_{alc}(A)} \subseteq \sigma_{alc}(T(t)).$

Lemma 3.2. *Let $(T(t))_{t \geq 0}$ a C_0 -semigroup, A its generator and for all $x \in X$, $B_\lambda(t)x = \int_0^t e^{\lambda(t-s)}T(s)xds$, $F_\lambda(t)x = \int_0^t e^{-\lambda s}B_\lambda(t)xds$. Let $u \in D(A), v \in X$ such that $(\lambda - A)u = B_\lambda(t)v$. Then there exists $w \in D(A)$ such that:*

- (1) $(\lambda - A)w = v;$
- (2) $B_\lambda(t)w = u;$
- (3) $\|w\| \leq (\|F_\lambda(t)\| + \|G_\lambda(t)\|) \max(\|u\|, \|v\|).$

Proof. Let $w = F_\lambda(t)v + G_\lambda(t)u \in D(A)$, then $\|w\| \leq (\|F_\lambda(t)\| + \|G_\lambda(t)\|) \max\{\|u\|, \|v\|\}$. And we have:

$$\begin{aligned}
 (\lambda - A)w &= (\lambda - A)F_\lambda(t)v + (\lambda - A)G_\lambda(t)u \\
 &= (I - B_\lambda(t)G_\lambda(t))v + (\lambda - A)G_\lambda(t)u \text{ by (Lemma 3.1 (2))} \\
 &= v - G_\lambda(t)B_\lambda(t)v + G_\lambda(t)(\lambda - A)u \\
 &= v
 \end{aligned}$$

$$\begin{aligned}
 B_\lambda(t)w &= B_\lambda(t)G_\lambda(t)u + B_\lambda(t)F_\lambda(t)v \\
 &= (I - (\lambda - A)F_\lambda(t))u + B_\lambda(t)F_\lambda(t)v \text{ by (Lemma 3.1 (2))} \\
 &= u - F_\lambda(t)(\lambda - A)u + F_\lambda(t)B_\lambda(t)v \\
 &= u
 \end{aligned}$$

□

Lemma 3.3. *Let $(T(t))_{t \geq 0}$ a C_0 -semigroup, with generator A . Then:*

- (1) $K(B_\lambda(t)) \cap K(\lambda - A) \subseteq K(e^{\lambda t} - T(t))$
- (2) $C(B_\lambda(t)) \cap C(\lambda - A) \subseteq C(e^{\lambda t} - T(t))$
- (3) $S(B_\lambda(t)) \cap S(\lambda - A) \subseteq S(e^{\lambda t} - T(t))$

Proof. 1): We have $K(B_\lambda(t)) \cap K(\lambda - A) \subseteq K(e^{\lambda t} - T(t))$.

Indeed : Let $x \in K(B_\lambda(t)) \cap K(\lambda - A)$. Then there exists $(x_{i,0})_{i \geq 0} \in D(A)$, $(x_{0,j})_{j \geq 0} \in X$ and $\delta > 0$ such that $(\lambda - A)x_{i,0} = x_{i-1,0}$, $\|x_{i,0}\| \leq \delta^i \|x\|$ and $B_\lambda(t)x_{0,j} = x_{0,j-1}$, $\|x_{0,j}\| \leq \delta^j \|x\|$.

We have $(\lambda - A)x_{1,0} = x_{0,0} = B_\lambda(t)x_{0,1}$, according to lemma 3.2 there exists a $x_{1,1} \in X$ such that :

$$(\lambda - A)x_{1,1} = x_{0,1} \text{ and } B_\lambda(t)x_{1,1} = x_{1,0}$$

and we have $(\lambda - A)x_{2,0} = x_{1,0} = B_\lambda(t)x_{1,1}$, lemma 3.2 implies there exists a $x_{2,1}$ such that $(\lambda - A)x_{2,1} = x_{1,1}$, hence $B_\lambda(t)x_{0,2} = x_{0,1} = (\lambda - A)x_{1,1}$ consequently there exists $x_{1,2}$ such that $B_\lambda(t)x_{1,2} = x_{1,1}$, therefore $(\lambda - A)x_{2,1} = B_\lambda(t)x_{1,2} = x_{1,1}$, lemma 3.2 implies there exists $x_{2,2}$ such that

$$B_\lambda(t)x_{2,2} = x_{2,1} \text{ and } (\lambda - A)x_{2,2} = x_{1,2}$$

Let $x_{2,2} = F_\lambda(t)x_{1,2} + G_\lambda(t)x_{2,1}$, by induction we can construct a sequence $(x_{i,j})_{i,j \geq 0}$ defined by $x_{i,j} = F_\lambda(t)x_{i-1,j} + G_\lambda(t)x_{i,j-1}$, and we have :

$$(\lambda - A)x_{i,j} = x_{i-1,j} \text{ and } B_\lambda(t)x_{i,j} = x_{i,j-1}$$

and $\|x_{i,j}\| \leq \delta \max(\|F_\lambda(t)\| + \|G_\lambda(t)\|)^{i+j} \|x\|$, for all $i, j \geq 1$

Let $y_i = x_{i,i}$, then $y_0 = x_{0,0} = x$ and

$$\begin{aligned} (e^{\lambda t} - T(t))y_i &= (\lambda - A)B_\lambda(t)y_i \\ &= (\lambda - A)B_\lambda(t)x_{i,i} \\ &= x_{i-1,j-1} = y_{i-1} \end{aligned}$$

and $\|y_i\| = \|x_{i,i}\| \leq \beta^i \|x\|$ where $\beta > 0$, hence $x \in K(e^{\lambda t} - T(t))$.

2): Similar to 1)

3): Let $x \in S(B_\lambda(t)) \cap S(\lambda - A)$. Then there exists $(x_{i,0})_{i \geq 0} \in D(A)$, $(x_{0,j})_{j \geq 0} \in X$, $x_{0,0} = 0$, such that $(\lambda - A)x_{i,0} = x_{i-1,0}$, $\sup_i \|x_{i,0}\|^{\frac{1}{i}} < \infty$ and $B_\lambda(t)x_{0,j} = x_{0,j-1}$, $\sup_j \|x_{0,j}\|^{\frac{1}{j}} < \infty$, by the same arguments in 1), we can show that $x \in S(e^{\lambda t} - T(t))$. \square

Theorem 3.3. *Let $(T(t))_{t \geq 0}$ a C_0 -semigroup, with generator A . Then*

- (1) $K(e^{\lambda t} - T(t)) \cap K(B_\lambda(t)) = K(\lambda - A) \cap K(B_\lambda(t))$,
- (2) $C(e^{\lambda t} - T(t)) \cap C(B_\lambda(t)) = C(\lambda - A) \cap C(B_\lambda(t))$.

Proof. Corollary 3.3 and lemma 3.3 gives the result. \square

4. STABILITY RESULTS.

Let $(T(t))_{t \geq 0}$ a C_0 -semigroup on X with infinitesimal generator A . $\{T(t)\}_{t \geq 0}$ is said to be strongly stable if $\lim_{t \rightarrow \infty} \|T(t)x\| = 0$ for all $x \in X$. We say that $(T(t))_{t \geq 0}$ is uniformly stable if $\lim_{t \rightarrow \infty} \|T(t)\| = 0$.

In [2], A. Elkoutri and M. A. Taoudi showed that $(T(t))_{t \geq 0}$ is strongly stable if $\sigma_K(A) \cap i\mathbb{R} = \emptyset$.

In the following, we give a stability result for strongly continuous semigroups using the local spectrum:

Theorem 4.1. *Let A be the generator of a bounded strongly continuous semigroup $(T(t))_{t \geq 0}$. If for all $x \in X$, $\sigma_A(x) \cap i\mathbb{R} = \emptyset$, then $(T(t))_{t \geq 0}$ is strongly stable.*

Proof. If $\sigma_A(x) \cap i\mathbb{R} = \emptyset$ for all $x \in X$, then

$$\emptyset = \bigcup_{x \in X} (\sigma_A(x) \cap i\mathbb{R}) = \bigcup_{x \in X} \sigma_A(x) \cap i\mathbb{R} = \sigma_{su}(A) \cap i\mathbb{R}.$$

As $\sigma_K(A) \cap i\mathbb{R} \subseteq \sigma_{su}(A) \cap i\mathbb{R} = \emptyset$, then $\sigma_K(A) \cap i\mathbb{R} = \emptyset$. According to [2, corollary 2.1], $(T(t))_{t \geq 0}$ is strongly stable \square

Theorem 4.2. *Let A be the generator of a bounded strongly continuous semigroup $(T(t))_{t \geq 0}$. Then, the following assertions are equivalent:*

- (1) $(T(t))_{t \geq 0}$ is uniformly stable;
- (2) for all $x \in X$, there exist $t_0 > 0$ such that $\sigma_{T(t_0)}(x) \cap \Gamma = \emptyset$

where Γ stands for the unit circle of \mathbb{C} .

Proof. According to [2, corollary 2.2] and [3, Theorem 3.2], it suffices to show that $\sigma_{T(t_0)}(x) \cap \Gamma = \emptyset$ implies that $\sigma_K(T(t_0)) \cap \Gamma = \emptyset$. Indeed: If $\sigma_{T(t_0)}(x) \cap \Gamma = \emptyset$ for all $x \in X$, then

$$\emptyset = \bigcup_{x \in X} (\sigma_{T(t_0)}(x) \cap \Gamma) = \bigcup_{x \in X} \sigma_{T(t_0)}(x) \cap \Gamma = \sigma_{su}(T(t_0)) \cap \Gamma.$$

As $\sigma_K(T(t_0)) \cap \Gamma \subseteq \sigma_{su}(T(t_0)) \cap \Gamma = \emptyset$, then $\sigma_K(T(t_0)) \cap \Gamma = \emptyset$.

□

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A. TAJMOUATI, M. KARMOUNI, H. BOUA, Z. AL-HOMIDI
 SIDI MOHAMED BEN ABDELLAH UNIVERISTY FACULTY OF SCIENCES DHAR AL MAHRAZ FEZ, MOROCCO.

E-mail address: `abdelaziztajmouati@yahoo.fr`

E-mail address: `mohammed.karmouni@usmba.ac.ma`

E-mail address: `hamid12boua@yahoo.com`

E-mail address: `zakariya1978@yahoo.com`